Divisorial Adjunction

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Setting: $(X', B'+M')$ generalized pair.

S' = Normalization of component of B' w/ coeff $= 1$

Then, can get generalized pair $(S', B_{S'} + M_{S'})$ s.t. :

$$K_{S'} + B_{S'} + M_{S'} \sim_{\mathbb{R}} (K_{X'} + B' + M')|_{S'}$$

Facts: (1) $(X', B'+M')$ is generalized lc $\Rightarrow$ $(S', B_{S'} + M_{S'})$ is generalized lc.

(2) (Generalized inversion of adjunction)

Assume (X', S') is plt. Then:

$(S', B_{S'} + M_{S'})$ generalized lc $\Rightarrow$ $(X', B'+M')$ generalized lc near S'.

Adjunction for fiber spaces

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Similar to the canonical bundle formula.

Setting:
- (X,B) projective sub-pair
- $X \xrightarrow{f} Z$ contraction w/ (X,B) sub-lc near generic fiber of f
- $K_X + B \sim_{\mathbb{R}} 0/Z$

Defn: Define a divisor B_Z on Z as follows:

$t_D :=$ lct of f^*D wrt (X,B) over generic pt. of D

$B_Z = \Sigma(1-t_D)D$ ("discriminant part")

Define M_Z ("moduli part") by the equation:

$$K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z)$$

Facts: ① If we have:

$$\begin{tikzcd} X' \arrow[r, "\phi"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Z' \arrow[r, "\psi", "\text{bir}"'] & Z \end{tikzcd}$$

w/ $K_{X'} + B' = \phi^*(K_X + B)$

Then $\psi_* B_{Z'} = B_Z$

Can choose $M_{Z'}$ s.t. $\psi_* M_{Z'} = M_Z$

② Assume (X,B) is lc near the generic fiber of f

For a suitable resolution $Z' \to Z$, $M_{Z'}$ is pseudoeffective.

Lem 3.7: Fix $\varepsilon \in \mathbb{R}$.

Suppose $\exists$ prime divisor S on some bir. model over X s.t.:
- $a(S; X, B) \le \varepsilon$.
- S is vertical over Z.

Then there is a resolution $Z' \to Z$ and a component T of $B_{Z'}$ with coeff $\geq 1-\varepsilon$

"(X,B) not ε-lc $\Rightarrow$ $(Z', B_{Z'})$ not ε-lc"

Adjunction on non-klt centers

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Setting:
- (X,B) proj. klt pair of dim d.
- $G \subseteq X$, $F \to G$ normalization
- X = $\mathbb{Q}$-factorial near gen. pt. of G
- Δ = $\mathbb{R}$-Cartier on X, ≥ 0
- $(X, B+\Delta)$ = lc near gen. pt. of G.

  $\exists$ unique non-klt place of $(X, B+\Delta)$ whose center is G

Goal: Want to do adjunction to G (rather F).

We will define $\mathbb{R}$-divisor Θ_F on F with coeff $\in [0,1]$.

Gives:

$$K_F + \Theta_F + P_F \sim_{\mathbb{R}} (K_X + B + \Delta)|_F$$

Idea of defn. Θ_F: Extract a suitable divisor S over X, which maps to F, $S \to F$

Θ_F is just the "discriminant part" associated to $(S, \mathcal{E}_S) \to F$

Definition of Θ_F

① Extracting S:

- Set $\Gamma = (B+\Delta)^{<1} + \mathrm{Supp}(B+\Delta)^{\geq 1}$

  $N = (B+\Delta) - \Gamma \to$ Components $\subseteq$ non lc loci.

- Let $W \xrightarrow{\phi} X$ log res. of $(X, B+\Delta)$

  Set $\Gamma_W = \widetilde{\Gamma} + \mathrm{Exc}(\phi)$

  $N_W = \phi^*(K_X + B + \Delta) - (K_W + \Gamma_W)$

  Clearly $\phi_* N_W = N$.

  N_W^+ is made up of all the divisors with log discrep < 0.

  N_W^- is the exceptional, log discrep > 0 part.

  Will run MMP to get rid of N_W^-.

- Run a $K_W + \Gamma_W$ - MMP$/X$ w/ scaling of some ample divisor.

  Get Y where $K_Y + \Gamma_Y$ is a limit of movable $/X$ $\mathbb{R}$-divisors (2.9).

- General negativity lemma $\Rightarrow$ $N_Y \geq 0$!

  Got rid of negative part!

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  Got rid of negative part !

- Letting $U =$ Locus where $(X, B+\Delta)$ is lc,
  $\Rightarrow N_Y$ lives over $X \setminus U$. [In particular $G \not\subseteq \mathrm{Image}(N_Y)$]
  $\therefore (Y, \Gamma_Y)$ is $\mathbb{Q}$-factorial dlt model of $(X, B+\Delta)$ over U.

  Lemma 2.33 $\Rightarrow$ $\exists$ unique component S of $\lfloor \Gamma_Y \rfloor$ mapping onto G.
  Get $S \xrightarrow{h} F$ contraction.

  Rmk: Divisorial adjunction gives:
$$K_S + \Gamma_S + N_S = (K_Y + \Gamma_Y + N_Y)|_S$$
  where $\Gamma_S = (\Gamma_Y - S)|_S$
  $N_S = N_Y|_S$

② Defining the boundary Σ_S:

- If codim $G > 1$: Let $\Sigma_Y = \mathrm{Exc}/_X + \widetilde{B}$
- If codim $G = 1$: Let $\Sigma_Y = \mathrm{Exc}/_X + \widetilde{B} + (1 - \mu_G B) S$ $\rightarrow$ This is just to ensure coeff of S in Σ_Y is 1.

Either case, S is a component of Σ_Y with coeff 1.
Divisorial adjunction gives:
$$K_S + \Sigma_S = (K_Y + \Sigma_Y)|_S$$

Rmk: $\Sigma_Y \leq \Gamma_Y$
$\therefore \Sigma_S \leq \Gamma_S$

③ Defining Θ_F: $\Theta_F :=$ "Discriminant part" associated to $(S, \Sigma_S) \xrightarrow{h} F$ i.e.
$$\Theta_F = \Sigma (1 - t_D) D$$
where $t_D = \mathrm{lct}$ of $h^* D$ w.r.t. (S, Σ_S) over the generic pt. of D.

Thm 3.10: Let $\Phi \subseteq [0,1]$ subset containing 1.
Let $\mathrm{coeff}(B) \subset \Phi$
Then: • $\mathrm{coeff}(\Theta_F) \in \Psi := \{a \mid 1 - a \in LCT_{d-1}(D(\Phi))\} \cup \{1\}$
• P_F is pseudoeff.

Pf: • $\mathrm{coeff}(\Sigma_S) \in D(\Phi)$

$\Rightarrow$ coeff $(\Theta_F) \subseteq 1 - \mathrm{lct}(D(\Phi))$

- $$\begin{tikzcd} S \arrow[r] \arrow[d, "h"'] & Y \arrow[d, "\pi"] \\ F \arrow[r] & X \end{tikzcd}$$

$$K_S + \Gamma_S + N_S \sim (K_Y + \Gamma_Y + N_Y)|_S \sim (\pi^*(K_X + B + \Delta))|_S$$
$$\wr \qquad\qquad \sim h^*((K_X + B + \Delta)|_F)$$
$$h^*(K_F + \Delta_F + R_F) \qquad \sim h^*(K_F + \Theta_F + P_F)$$

- Let Δ_F, R_F be the discriminant, moduli parts of adjunction for $\underbrace{(S, \Gamma_S + N_S) \to F}$

$\Rightarrow$ $\underline{\Delta_F + R_F} \sim_{\mathbb{R}} \underline{\Theta_F + P_F}$

Recall $\Sigma_S \leq \Gamma_S$ $\therefore$ $\Sigma_S \leq \Gamma_S + N_S$

$\Rightarrow$ $\Theta_F \leq \Delta_F$

$\therefore$ $P_F - R_F \geq 0$.

Since R_F pseudoeff (3.6) $\Rightarrow$ P_F pseudoeff.

Lem 3.12: Assume • G = general member of a covering family of subvarieties of X

[i.e. $\exists$ family of subvarieties $V \xrightarrow{f} T$

Have $\begin{tikzcd} V \arrow[r] \arrow[d, "f"] & X \\ T & \end{tikzcd}$ s.t. • V, T proj. varieties, f contraction.

• $V_t \hookrightarrow X$ is a subvariety $\forall t \in T$

• $V \to X$ is surjective

• G is a general fiber of f]

• (X, B) is ε-lc for some $\varepsilon > 0$.

Then $\exists$ sub-boundary B_F on F s.t. • $K_F + B_F = (K_X + B)|_F$

• (F, B_F) is sub-ε-lc

• $B_F \leq \Theta_F$

Pf: ① Get new family $W' \to R'$ s.t. $W' \to X$ is generically finite

- Normalize, take resolutions of V, T to get V', T' smooth proj. varieties

Can also assume $\exists$ Cartier div $P \geq 0$ on X s.t. • Supp $B \cup X_{\mathrm{sing}} \subseteq$ Supp P

• $Q' :=$ Pullback of P to V' is relatively snc over some open subset of T'.

- Call F' the fiber corresponding to F.

$$\begin{tikzcd} F' \arrow[d] & \subseteq & V' \arrow[r, "g'"] \arrow[d, "f'"] & X \\ t' & \in & T' & \end{tikzcd}$$

Since "general", can assume: • f' smooth over t'

• $g'(\eta_{F'})$ smooth pt of X

• g' smooth over $g'(\eta_{F'})$

• 2.28 ⇒ Cut T by general hyperplane sections to get:

$$\begin{matrix} W' & \xrightarrow{\text{gen. finite}} & X \\ \downarrow & & \\ R' & & \end{matrix} \quad + \text{ everything as before also holds.}$$

Let $Q_{W'} := Q'|_{W'}$. [Observe $Q_{W'}|_{F'}$ is reduced, snc.

∴ Near F', $Q_{W'}$ is reduced.

⇒ Any prime divisor C on F' is contained in at most one comp. of $Q_{W'}$]

② Define B_F on F and show (F, B_F) sub-ε-lc

• Define $B_{W'}$ by $K_{W'} + B_{W'}$ = Pullback of $K_X + B$

$$\begin{matrix} W' & \xrightarrow{\text{gen-finite}} & X \\ \downarrow & & \\ R' & & \end{matrix}$$

Stein factorization + behavior under finite maps ⇒ $\underline{(W', B_{W'})}$ is sub-ε-lc

Define $\underline{B_{F'}} := B_{W'}|_{F'}$.

Observe $K_{F'} = K_{W'}|_{F'}$

∴ $K_{F'} + B_{F'} = (K_{W'} + B_{W'})|_{F'}$

Define B_F via pushforward to F:

⇒ $K_F + B_F = (K_X + B_X)|_F$

• To show (F, B_F) is sub-ε-lc, suffices to show $(F', B_{F'})$ is sub-ε-lc.

, suffices to show $(F', B^+_{F'})$ is sub-ε-lc

(where $B^+_{F'} = B^+_{W'}|_{F'}$)

But $B^+_{W'} \subseteq \operatorname{Supp} Q_{W'}$ by construction

⇒ $B^+_{F'} \subseteq \operatorname{Supp} Q_{F'}$

⇒ $B^+_{F'}$ snc. ∴ Suffices to show coeff of comp. of $B^+_{F'} \le 1-\varepsilon$

Coeff of comp. of $B^+_{F'}$ = Coeff of comp of $B^+_{W'}$

∴ $(F', B^+_{F'})$ sub-ε-lc since $(W', B^+_{W'})$ sub-ε-lc.

③ Additional setup

For the rest of the proof, fix prime divisor C on F.

We will prove that $\mu_C B_F \le \mu_C \Theta_F$. And so, $B_F \le \Theta_F$ as we vary over all C.

- If $\mu_C B_F \leq 0$, there is nothing to check

  Assume $\mu_C B_F > 0$.

  Let C' = corresponding to C on F'.

  D' = Unique component of $B_{W'}^{+}$ s.t. $C' \subseteq D'|_{F'}$.

  $\therefore \mu_{C'} B_{F'} \leq \mu_{D'} B_{W'}$

④ <u>Compute $\mu_C \Theta_F$, other lcts</u>

- Set $L = \lambda P$ s.t. $\mu_{D'} L_{W'} = 1$

  ($\Leftrightarrow \mu_{C'} L_{F'} = 1$)

  $\therefore L_F$ looks like C near the gen. pt. of C.

  Let t: lct of L_S w.r.t. (S, Σ_S) over the gen. pt of C

  Then $\mu_C \Theta_F = 1 - t$

- Let s = lct of L wrt. (X, B) at gen. pt. of C

  I = minimal non-klt center of $(X, B + sL)$ which contains gen. pt. of C

  I_Y = non-klt center of $(Y, B_Y + sL_Y)$ which maps to I.

  $I_Y \neq S$

  As $\Sigma_Y \geq B_Y$, I_Y is also non-klt center of $(Y, \Sigma_Y + sL_Y)$.

⑤ <u>Show $t \leq s$ (Assuming X = $\mathbb{Q}$-factorial)</u>

Suffices to prove some non-klt center of $(S, \Sigma_S + sL_S)$ maps to C.

We show $I_Y \cap S$ is a non-klt center of $(S, \Sigma_S + sL_S)$ which maps to C

⑥ <u>Show $\mu_C B_F \leq \mu_C \Theta_F$</u>

$(X, B + sL)$ lc near gen. pt. of $C \Rightarrow (W', B_{W'} + sL_{W'})$ lc over gen. pt. of C

$\therefore \mu_{D'} B_{W'} + s \leq 1$

Now:

$$\mu_C B_F + t = \mu_{C'} B_{F'} + t \leq \mu_{C'} B_{F'} + s \leq \mu_{D'} B_{W'} + s \leq 1$$

$$\Rightarrow \mu_C B_F \leq 1 - t = \mu_C \Theta_F$$

⑦ <u>Remove $\mathbb{Q}$-factorial assumption.</u>

Take small $\mathbb{Q}$-factorialization $\overline{X} \to X$

$Y \to X$, $W' \to X$ can be made to factor through $\overline{X} \to X$.

Thus, the pushforwards of $\Theta_{\overline{F}}$, $B_{\overline{F}}$ to F are Θ_F, B_F.

Thus $B_{\overline{F}} \leq \Theta_{\overline{F}} \Rightarrow B_F \leq \Theta_F$.

$\blacksquare$

Lifting sections from non-klt centers

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Goal: Under certain assumptions, can lift sections from non-klt centers.

Lem 3.14: Assume P_F is big.

Then: ① $(S, \Gamma_S + N_S)$ not ε-lc + center is vertical over F $\Rightarrow$ $(F, \Theta_F + P_F)$ not ε-lc

② Fix $\delta \geq 0$.

$(X, B+\Delta)$ has non-klt center H ($\neq G$) intersecting G $\Rightarrow$ $(F, \Theta_F + P_F)$ not δ-lc

Pf: Let $\Delta_Y = \Gamma_Y + N_Y$, $\Delta_S = \Gamma_S + N_S$

① Lemma 3.7 + Certain reductions $\Rightarrow$ Comp. T of Δ_F has coeff $\geq 1-\varepsilon$

where Δ_F, R_F = Discriminant, moduli parts of $(S, \Delta_S) \to F$

Pick t s.t. $\varepsilon > t > 0$.

Want to prove $(F, \Theta_F + P_F)$ is not ε-lc

$$\Theta_F + P_F \sim t\Theta_F + \underbrace{t P_F}_{\text{big}} + \underbrace{(1-t) R_F}_{\text{pseudoeff}} + \underbrace{(1-t)\Delta_F}_{\text{comp. with coeff} \geq 1-\varepsilon}$$

$\Theta_F + P_F \sim \Delta_F + R_F$

≥ 0 $\quad \geq 0$

$\therefore$ Can choose P_F s.t. $(F, \Theta_F + P_F)$ is not ε-lc.

② $\exists$ non-klt center $Z \neq S$ of (Y, Δ_Y) intersecting S (by connectedness principle).

Since (Y, Γ_Y) is dlt and Supp $N_Y \subseteq \lfloor \Gamma_Y \rfloor$,

Non-klt locus of $(Y, \Delta_Y) = \lfloor \Delta_Y \rfloor$

$\therefore$ Some component of $\lfloor \Delta_Y - S \rfloor$ intersects S - Comp has log disc $\leq 0 < \delta$

$\rightsquigarrow$ Get comp. of $\lfloor \Delta_S \rfloor$ which is vertical over F

& log discrep of comp. $= 0 < \delta$

① $\Rightarrow$ $(F, \Theta_F + P_F)$ not δ-lc. ▨

Prop 3.15: (Lifting sections from non-klt centers)

Fix $d, r \in \mathbb{N}$, $\varepsilon \geq 0$.

Then $\exists\, l = l(d, r, \varepsilon) \in \mathbb{N}$ satisfying the following.

Assume:
- X = Fano of dim d. $B = 0$
- G: general member of a covering family of subvar.
- $\Delta \sim_{\mathbb{Q}} -(n+1)K_X$ for some $n \in \mathbb{N}$

- $h^0(-nr K_X|_F) \neq 0$
- P_F is big and for any choice of $P_F \geq 0$ in its $\mathbb{R}$-linear equivalence class, $(F, P_F + \Theta_F)$ is ε-lc.

Then $h^0(-lnr K_X) \neq 0$.

Pf: ① $\underline{G \text{ is an isolated non-klt center}}$

If not, $\exists$ non-klt center $H \neq G$ intersecting G.

Lem 3.14 $\Rightarrow$ Can choose $P_F \geq 0$ s.t. $(F, \Theta_F + P_F)$ is not ε-lc. Contradiction!

$\therefore \nexists$ non-klt center intersecting G.

In particular, no comp. of $\lfloor \Delta_Y - S \rfloor$ intersects S.

(eg. each exc. div$/X$ does not intersect S)

$\pi: Y \to X$ as before.

② $\underline{E := \pi^*(-nr K_X) \text{ is integral near } S; \text{ has bounded Cartier index near codim 1 pts. of } S}$

- Comp. of E which are not integral = All exceptional over X.

  ① $\Rightarrow$ None of these intersect S.

  $\therefore$ To prove E integral near S, only need to check coeff of S in E.

  Generic pt. of G is a smooth pt. of X

  $\Rightarrow K_X$ is Cartier near this pt.

  $\Rightarrow$ coeff of S in E is integral.

- Let V = prime divisor on S

  Need to prove E has bounded Cartier index near generic pt. of V.

  - V horizontal over $G \Rightarrow$ gen. pt of V maps to gen. pt. of G

    $\Rightarrow E$ Cartier near gen. pt. of V

  - V vertical over $G \Rightarrow$ Let p be the Cartier index of $K_Y + S$ near gen. pt. of V.

    $\xRightarrow{\text{Shokurov}} \mu_V \Delta_S \geq 1 - \frac{1}{p}$ and Cartier index of E divides p.

    If $\frac{1}{p} < \varepsilon$, (S, Δ_S) not ε-lc + V vertical $\Rightarrow (F, \Theta_F + P_F)$ not ε-lc

    Contradiction!

    $\therefore \frac{1}{p} \geq \varepsilon \Rightarrow p \leq \frac{1}{\varepsilon}$

    $\Rightarrow$ Cartier index of E bounded! $(\leq \frac{1}{\varepsilon})$

③ $\underline{\text{KV vanishing} \Rightarrow \text{For } l \geq 2,\ h^1(\lceil lE - \lfloor \Gamma_Y \rfloor - N_Y \rceil) = 0}$

Define L as:

Define $\mathcal{L}$ as:

$$\lceil lE - \lfloor \Gamma_Y \rfloor - N_Y \rceil = lE - \lfloor \Gamma_Y \rfloor - N_Y + \mathcal{L}$$

$$= \underbrace{\underbrace{lE}_{\pi^*(-lnr K_X)} - \underbrace{(K_Y + \Delta_Y)}_{\pi^*(K_X+\Delta) = \pi^*(-nK_X)}}_{\text{big and nef}} + \underbrace{K_Y + \Delta_Y - \lfloor \Gamma_Y \rfloor - N_Y + \mathcal{L}}_{\underbrace{K_Y}_{} + \underbrace{\Gamma_Y - \lfloor \Gamma_Y \rfloor + \mathcal{L}}_{\text{klt}}}$$

$\therefore$ KV vanishing $\Rightarrow$ $\underline{h^1(\lceil lE - \lfloor \Gamma_Y \rfloor - N_Y \rceil) = 0}$

④ Lift sections to Y

Have $0 \to \mathcal{O}_X(-S) \to \mathcal{O}_X \to \mathcal{O}_S \to 0$

$\otimes$ by $\mathcal{O}_X(\lceil lE - \underbrace{\lfloor \Gamma_Y \rfloor - N_Y + S}_{F} \rceil)$

Hope to get:

$$0 \to \mathcal{O}_X(\lceil lE - \lfloor \Gamma_Y \rfloor - N_Y \rceil) \to \mathcal{O}_X(\lceil lE - \lfloor \Gamma_Y \rfloor - N_Y + S \rceil) \to \mathcal{O}_S((\quad)|_S) \to 0$$

$$\mathcal{O}_S((\quad)|_S) = \mathcal{O}_S(\lceil lE \rceil |_S)$$

But this is true only if: letting $U =$ Largest open set where F is Cartier.
Then $\underbrace{S \setminus S \cap U \text{ is of codim} \geq 2}$.

(Lem 2.42)

$\lceil lE - \underbrace{\lfloor \Gamma_Y \rfloor - N_Y + S}_{} \rceil$ near S $=$ $\lceil lE \rceil$

Choose l large enough, $\lceil lE \rceil$ is Cartier near codim 1 pts of S

$$h^0(\lceil lE - \lfloor \Gamma_Y \rfloor - N_Y + S \rceil) \twoheadrightarrow h^0(lE|_S)$$

$$\neq 0 \qquad \leq 0 \qquad\qquad \neq 0$$

$\Rightarrow h^0(\lceil lE \rceil) \neq 0$

$\Rightarrow h^0(-lnr K_X) \neq 0$ $\quad$ l bounded. ▨